Received: *March 4, 1999*, Accepted: *May 6, 1999*

A note on heterotic/type I′ duality and D0 brane quantum mechanics

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Abstract: In the note a simple calculation of one loop threshold corrections for the SO(32) heterotic string is performed. In particular the compactification on T^2 with a Wilson line breaking the gauge group to SO(16) × SO(16) is considered. Using heterotic type I duality, these corrections can be related to quantities appearing in the quantum mechanics of type I' D0 particles.

Keywords: Superstrings and Heterotic Strings, D-branes, String Duality, M(atrix) Theories.

JHEP05(1999)007

Contents

1. Introduction

The quantum mechanics of D0 branes and coming with it the question of existence of bound states of D0 branes is important for various string dualities. M-theory/type IIA duality implies that there is a single bound state of N D0 branes for any N corresponding to the Kaluza-Klein modes on the M-theory circle [1]. The relevant index for the case of two D0 branes was computed in [2, 3]. In this calculation the index is split into a bulk and boundary term which in turn is expressed as a zero dimensional matrix integral, whose value for arbitrary N was conjectured in [4] using results from [5]. The bulk term of the index was directly calculated in [6] using methods of topological field theory.

Another interesting example quantum mechanics of D0 particles appears in the so called type I' theory, which describes D0 particles in the presence of D8 branes and orientifold O8 planes [7, 8, 9, 10]. This theory is important for the matrix theory formulation of the heterotic string [11, 12, 13, 14]. Type I′ string theory is defined as the orientifold on IIA on $S^1/\mathbb{Z}_2$ which is T-dual to type I theory. There are two orientifold eight planes at the two ends of the interval and sixteen D8 branes in between. If eight D8 branes are on top of each orientifold plane, the gauge symmetry is given by SO(16) $\times$ SO(16). The strong coupling limit of this system is given by M-theory on $S^1/\mathbb{Z}_2$ [15]. The states which are not present in perturbative type I$'$ spectrum, but which are needed to fill out the $E_8 \times E_8$ multiplets, are given by bound states of D0-particles [16, 17]. Hence we have to look for bound states of D0-particles transforming in the **128** and **120** of SO(16).

It is interesting to find the description of these states in the heterotic SO(32) theory where they are perturbatively realized. These states are BPS states or which $N_R = 0(1/2)$ or the R(NS) sector and the mass and level matching [10, 16, 18] conditions become

$$\frac{p_R^2}{2} = \frac{p_L^2}{2} + N_L - 1\,, \qquad m^2 = p_R^2\,, \tag{1.1}$$

where the right moving momenta in $\Gamma_{17,1}$ are given by

$$\begin{aligned} p_L &= \left(P + Yn, \frac{m - \frac{1}{2}Y^2 n - YP}{2R} - nR\right) \\ p_R &= \frac{m - \frac{1}{2}Y^2 n - YP}{2R} + nR\,, \end{aligned} \tag{1.2}$$

where Y is a Wilson line along S^1 and P are momenta in SO(32) lattice and m and N are the momentum and the winding along S^1 respectively. With a Wilson line given by $Y = (0^8, (1/2)^8)$, SO(32) is broken to SO(16) $\times$ SO(16) and the analysis in [16] shows that the states with $N_L = 0$ and even n lie in the $(\mathbf{120}, \mathbf{1}) + (\mathbf{1}, \mathbf{120})$ of SO(16) $\times$ SO(16) whereas the states with odd n lie in $(\mathbf{128}, \mathbf{1}) + (\mathbf{1}, \mathbf{128})$

2. Heterotic one loop thresholds

The duality of the SO(32) heterotic and type I strings in ten dimensions makes it possible to calculate some nonperturbative effects on the type I side due to Euclidean D-branes exactly by a one loop calculation on the heterotic side. The simplest case in which such a calculation is possible arises for the heterotic string compactified on a two torus T^2 [19, 20, 21, 22, 23]. Worldsheet instantons on the heterotic side get mapped to wrapped Euclidean D-branes on the torus, which provide D-instanton effects in eight dimensions.

There are one loop heterotic thresholds [24, 25] which are BPS-saturated and related by supersymmetry to anomaly canceling terms [26] and therefore presumably exact at one loop. The one loop integrals involved are almost holomorphic since only BPS-states run in the loop and they can be calculated exactly. For the SO(32) heterotic string the relevant loop amplitude for gravitational thresholds with Wilson lines is given by

$$I_d = -\mathcal{N}(2\pi)^d \int_F \frac{d^2\tau}{\tau_2^{2-d/2}} \Gamma_{d,d+16} A(R,\tau)\,, \tag{2.1}$$

with $\mathcal{N} = V^{(10-d)}/(2^{10}\pi^6)$. The lattice function $\Gamma_{d,d+16}(G,B,Y)$ is given by

$$\begin{aligned}\Gamma_{d,d+16} = {} & \frac{\sqrt{\det(G)}}{\tau_2^{d/2}} \sum_{m^i,n^i} e^{-\frac{\pi}{\tau_2}(G+B)_{ij}(n+m\tau)^i(n+m\bar\tau)^j} \times \\ & \times \sum_{a,b=0,1} \prod_{k=1}^{16} e^{-i\pi(m^i n^j Y_i^k Y_j^k + b m^i Y_i^k)} \theta \begin{bmatrix} a + 2m^l Y_l^k \\ b + 2n^l Y_l^k \end{bmatrix} (0,\tau)\,.\end{aligned} \tag{2.2}$$

Here Y_i^k $i = 1,\dots,16$, $k = 1,\dots,d$ parameterize the Wilson lines around the cycles of T^d. The almost holomorphic A is given by

$$A(R,\tau) = \frac{1}{2^7 3^2 5} \frac{E_4(\tau)}{\eta^{24}(\tau)} t_8 \mathrm{Tr} R^4 + \frac{1}{2^9 3^2} \frac{\hat{E}_2^2(\tau)}{\eta^{24}(\tau)} t_8 (\mathrm{Tr} R^2)^2\,, \tag{2.3}$$

where $E_{2n}(\tau)$ are the Eisenstein modular forms of weight $2n$.

3. Two torus compactification with Wilson lines

We are interested in the T^2 compactification with the Kähler and complex structure modulus T, U of the torus. The SO(32) gauge symmetry will be broken to SO(16) $\times$ SO(16) by introducing Wilson lines on the two torus of the following form

$$Y_i^1 = \left(0^8, \frac{1}{2}^8\right), \qquad Y_i^2 = (0^8, 0^8)\,. \tag{3.1}$$

This choice of Wilson lines corresponds in the type I$'$ picture to eight D8 branes sitting on top of each of the two O8 planes, canceling dilaton and Ramond-Ramond sources locally.

We want to calculate the one loop thresholds of the form $t_8\mathrm{Tr}(R^4)$, $t_8\mathrm{Tr}(F^4)_1$ and $t_8(\mathrm{Tr}(F^2)_1)^2$ in the presence of this Wilson line. The subscript on the field strength in the second and third term indicates that the trace is taken over the first SO(16) factor. The integrals that will appear in this calculations are of the following form

$$I_Q = \int_F \frac{d^2\tau}{\tau_2} \sum_A \frac{T_2}{\tau_2} \exp\left\{ 2\pi i T \det A - \frac{\pi T_2}{\tau_2 U_2} \left| \begin{pmatrix} 1 & U \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right\} QC(Y,A)\,. \tag{3.2}$$

Here the matrix A is given by 2×2 matrices with integer entries

$$A = \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix}, \qquad m_1, m_2, n_1, n_2 \in \mathbb{Z}, \tag{3.3}$$

and $C(Y,A)$ is the partition function of the SO(32) lattice which in general depend on the Wilson line Y and the matrix A and is given by

$$\begin{aligned} C(Y,A) &= \sum_{a,b=0,1} \prod_{k=1}^{16} e^{-i\pi(m^i n^j Y_i^k Y_j^k + b n^i Y_i^k)} \theta \begin{bmatrix} a + 2m^l Y_l^k \\ b + 2n^l Y_l^k \end{bmatrix} (0,\tau) \\ &= \sum_{a,b} \theta^8 \begin{bmatrix} a \\ b \end{bmatrix} (0,\tau) \theta^8 \begin{bmatrix} a + m_1 \\ b + n_1 \end{bmatrix} (0,\tau) . \end{aligned}$$

We introduced the standard notation for the theta functions

$$\theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \theta_1 , \qquad \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \theta_2 , \qquad \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \theta_3 , \qquad \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \theta_4 . \tag{3.4}$$

The form of the operator Q in (3.2) depends on the threshold in question. For gravitational thresholds $t_8 \mathrm{Tr}(R^4)$ and $t_8(\mathrm{Tr}(R^2))^2$ Q is independent of the spin structures and A in (3.4) and is given by (2.3).

The operator Q for $\mathrm{Tr}(F^4)$ and $(\mathrm{Tr}(F^2))^2$ can be found by 'gauging' (3.4), [24]. The Wilson line (3.1) breaks the gauge group to SO(16) $\times$ SO(16) and the thirty two free fermions of the SO(32) lattice are split into two sets of sixteen in (3.4). The result depends on the spin structures $[a,b]$ for the sixteen fermions which are associated with the first SO(16) in (3.4). For the $\mathrm{Tr}(F^4)$ threshold the operators are given by

$$\begin{aligned} Q_{\mathrm{Tr}(F^4)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) &= -\frac{1}{2^8 3} \theta_3^4 \theta_4^4(\tau) , \\ Q_{\mathrm{Tr}(F^4)} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) &= \frac{1}{2^8 3} \theta_2^4 \theta_4^4(\tau) , \\ Q_{\mathrm{Tr}(F^4)} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) &= -\frac{1}{2^8 3} \theta_2^4 \theta_3^4(\tau) , \end{aligned} \tag{3.5}$$

whereas for the $(\mathrm{Tr}(F^2))^2$ threshold the operator is given by

$$\begin{aligned} Q_{(\mathrm{Tr}(F^2))^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (\tau) &= \frac{1}{2^{10} 3^2} \left(e_2(\tau) + \hat{E}_2(\tau)\right)^2 , \\ Q_{(\mathrm{Tr}(F^2))^2} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\tau) &= \frac{1}{2^{10} 3^2} \left(e_3(\tau) + \hat{E}_2(\tau)\right)^2 , \\ Q_{(\mathrm{Tr}(F^2))^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\tau) &= \frac{1}{2^{10} 3^2} \left(e_4(\tau) + \hat{E}_2(\tau)\right)^2 . \end{aligned} \tag{3.6}$$

Where the following notation has been introduced

$$e_2 = \theta_3^4 + \theta_4^4 , \qquad e_3 = \theta_2^4 - \theta_4^4 , \qquad e_4 = -\theta_2^4 - \theta_3^4 , \tag{3.7}$$

and $\hat{E}_2$ is the nohomolomorphic (but modular) Eisenstein function of weight 2.

4. Evaluation of integral

The integral (3.2) can be evaluated using the method of orbits [27]. In the present context this technique was discussed in [19, 20] and in [21], where type I thresholds with certain Wilson lines present were evaluated using results from [28]. Without Wilson lines it is straightforward to show that under the modular $\mathrm{SL}(2,\mathbb{Z})$ transformations $\tilde{\tau} = (a\tau+b)/(c\tau+d)$ with $a,b,c,d \in \mathbb{Z}$,$ad-bc=1$

$$\frac{1}{\tilde{\tau}_2}\left|(1 \quad U\,)\,A\begin{pmatrix}\tilde{\tau}\\1\end{pmatrix}\right|^2 = \frac{1}{\tau_2}\left|(1U)A\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}\right|^2. \tag{4.1}$$

The summation over all integer matrices matrices A can then replaced by the summation over all equivalence classes of $\mathrm{SL}(2,\mathbb{Z})$ orbits. There are three different cases, the trivial orbit $A=0$, the degenerate orbit $\det(A)=0$ and the non degenerate orbit $\det(A)\neq 0$

In the following we will consider only the non degenerate orbit,where the fundamental $\mathcal{F}$ is unfolded into the double cover of the upper half plane $\mathcal{H}$. The non degenerate $\mathrm{SL}(2,\mathbb{Z})$ orbits fall into the following equivalence classes

$$A = \pm\begin{pmatrix}k & j\\0 & p\end{pmatrix}, \qquad k>0\,, \qquad 0\le j<k, p\in\mathbb{Z}\,. \tag{4.2}$$

When Wilson lines are present, matters are more complicated but using the well known transformation properties of the theta functions under $\tau\to\tau+1, \tau\to -1/\tau$ is is easy to see that for both $Q_{\mathrm{Tr}(F)^4}$ (3.5) and $Q_{(\mathrm{Tr}(F)^2)^2}$ (3.6), $QC(Y,A)$ defined in (3.4) behaves in the following way

$$QC(Y,A)\left(\frac{a\tau+b}{c\tau+d}\right) = QC\left(Y, A\begin{pmatrix}a & b\\c & d\end{pmatrix}\right)(\tau). \tag{4.3}$$

Hence the method of orbits can be used to unfold the integral. For the non degenerate orbit we get

$$I_{nd} = \int_H \frac{d^2\tau}{\tau_2}\sum_{\substack{k>0\\0\le j<2\\p\in\mathbb{Z}}}\frac{T_2}{\tau_2}\exp\left\{2\pi ikpT - \frac{\pi T_2}{\tau_2 U_2}\left|k\tau+j+pU\right|^2\right\} QC\left(Y,\begin{pmatrix}k & j\\0 & p\end{pmatrix}\right)(\tau). \tag{4.4}$$

In order to evaluate (4.4) it is convenient to split the summation over equivalence classes A in (4.2) into four seperate sectors $A^{(i)}, i=1,\ldots,4$.

$$A^{(1)} = \begin{pmatrix}2\tilde{k} & 2\tilde{j}\\0 & p\end{pmatrix}, \qquad 0\le 2\tilde{j}<2\tilde{k}\,,$$
$$A^{(2)} = \begin{pmatrix}2\tilde{k}+1 & 2\tilde{j}\\0 & p\end{pmatrix}, \qquad 0\le 2\tilde{j}<2\tilde{k}+1\,,$$

$$A^{(3)} = \begin{pmatrix} 2\tilde{k} & 2\tilde{j}+1 \\ 0 & p \end{pmatrix}, \qquad 0 \le 2\tilde{j}+1 < 2\tilde{k}\,,$$

$$A^{(4)} = \begin{pmatrix} 2\tilde{k}+1 & 2\tilde{j}+1 \\ 0p & \end{pmatrix}, \qquad 0 \le 2\tilde{j}+1 < 2\tilde{k}+1\,. \tag{4.5}$$

The expansion of $QC(Y, A^{(i)})$ appearing in (4.4) in powers of $q = \exp(2\pi i\tau)$ and powers of $1/\tau_2$ is given by

$$QC(Y, A^{(i)})(\tau) = \sum_{n\ge -1, r\ge 0} c^{(i)}_{n,r}\,\frac{1}{\tau_2^r} q^n\,. \tag{4.6}$$

The integral (4.4) is then of the form $I_{n,r}$ defined in the appendix. Such integrals were evaluated in [19, 20] and the main results are reviewed in the appendix for completeness.

The terms of order $1/q$ in (4.6) are problematic for the type I′ interpretation as discussed in section 6. For all $QC(A^{(i)})$ which will be considered later it turns out that only the $A^{(1)}$ and $A^{(3)}$ sector contribute terms of order $1/q$ in the integral. In addition we shall find that $c^{(1)}_{-1,r} = c^{(3)}_{-1,r}$. In this case the summation over $\tilde{j}$ of the two terms can be combined giving $\sum_{0\le j<2k} exp(-\pi j/k) = 0$ and hence these contribution vanish when summed over j.

In section 6 only terms of order q^0 in (4.6) will directly related to quantities in type I′ QM, which corresponds to taking the limit $U_2 \to \infty$. For these terms the nonholomorphic pieces in the $(\mathrm{Tr}(F^2))^2$ and $(\mathrm{Tr}(R^2))^2$ due to the presence of $\hat{E}_2$ will not supressed by inverse powers of U_2 in the $U_2 \to \infty$ limit as explained in the appendix.

Using (A.5) the $I_{0,0}$ part of the integral (4.4) can be expressed as,

$$I_{0,0} = \sum_{k,p}\Big\{ c^{(1)}_{0,0}\frac{1}{2|p|}e^{2\pi i 2kpT} + c^{(2)}_{0,0}\frac{k+1}{(2k+1)|p|}e^{2\pi i(2k+1)pT} + c^{(3)}_{0,0}\frac{1}{2|p|}e^{2\pi i 2kpT} +$$
$$+ c^{(4)}_{0,0}\frac{k}{(2k+1)|p|}e^{2\pi i(2k+1)pT}\Big\} + \mathrm{c.c.} \tag{4.7}$$

In all examples considered below we find that $c^{(2)}_{0,0} = c^{(4)}_{0,0}$, Hence the contributions of the $A^{(2)}$ and $A^{(4)}$ sector can be combined, rearranging the summation gives

$$I_{0,0} = \left(\frac{c^{(1)}_{0,0} + c^{(3)}_{0,0}}{2} - c^{(2)}_{0,0}\right)\sum_{N|n}\frac{1}{n}e^{2\pi i 2NT} + c^{(2)}_{0,0}\sum_{N|n}\frac{1}{n}e^{2\pi i NT} + \mathrm{c.c.} \tag{4.8}$$

Where $N|n$ denotes the set of all integers n which divide N.

4.1 $t_8\mathrm{Tr}(R^4)$ thresholds

For the $t_8\mathrm{Tr}(R^4)$ threshold, the operator Q does not depend on the spin structures of the theta functions associated with the first factor SO(16),

$$Q_{R^4} = \frac{1}{2^7 3^2 5}\frac{1}{\eta^{24}(\tau)}E_4(\tau)\,. \tag{4.9}$$

Combining (4.9) with (3.4) $QC(A^{(i)})$ for $\mathrm{Tr}(R^4)$ is given by

$$\begin{aligned}
QC(A^{(1)}) &= \frac{1}{2^7 3^2 5}\frac{E_4}{\eta^{24}}(\theta_2^{16}+\theta_3^{16}+\theta_4^{16}),\\
QC(A^{(2)}) &= \frac{1}{2^6 3^2 5}\frac{E_4}{\eta^{24}}\theta_2^8\theta_3^8\\
QC(A^{(3)}) &= \frac{1}{2^6 3^2 5}\frac{E_4}{\eta^{24}}\theta_3^8\theta_4^8,\\
QC(A^{(4)}) &= \frac{1}{2^6 3^2 5}\frac{E_4}{\eta^{24}}\theta_2^8\theta_4^8\,.
\end{aligned}\tag{4.10}$$

Expanding the terms in (4.10), confirms that $c^{(1)}_{-1,0} = c^{(3)}_{-1,0} = 1/2^6 3^2 5$ and $c^{(2)}_{-1,0} = c^{(4)}_{-1,0} = 0$ and hence terms of order $1/q$ do vanish in the integral after summation over j. Furthermore one finds $c^{(1)}_{0,0} = 744/2^6 3^2 5$, $c^{(2)}_{0,0} = c^{(4)}_{0,0} = 256/2^6 3^2 5$ and $c^{(3)}_{0,0} = 232/2^6 3^2 5$. Plugging these coefficients into (4.8) gives

$$I^{\mathrm{Tr}(R^4)}_{0,0} = \frac{1}{2^6 3^2 5}\left\{256\sum_{N|n}\frac{1}{n}e^{2\pi i NT} + 232\sum_{N|n}\frac{1}{n}e^{2\pi i 2NT}\right\}. \tag{4.11}$$

4.2 $t_8\mathrm{Tr}(F_1^4)$ thresholds

The operator Q for the threshold for $t_8\mathrm{Tr}(F_1^4)$ associated to a complex fermion with spin structure $[a,b]$ was defined in (3.5). Using (3.4) and (3.5) we can express $QC(A^{(i)})$ for the $\mathrm{Tr}(F_1)^4$ threshold as

$$\begin{aligned}
QC(A^{(1)}) &= \frac{1}{2^8 3}\frac{1}{\eta^{24}}(-\theta_2^{16}\theta_3^4\theta_4^4+\theta_3^{16}\theta_2^4\theta_4^4-\theta_4^{16}\theta_2^4\theta_3^4) = 1\,,\\
QC(A^{(2)}) &= \frac{1}{2^8 3}\frac{1}{\eta^{24}}\theta_2^8\theta_3^8(-\theta_3^4\theta_4^4+\theta_2^4\theta_4^4) = -\frac{1}{3}\,,\\
QC(A^{(3)}) &= \frac{1}{2^8 3}\frac{1}{\eta^{24}}\theta_3^8\theta_4^8(\theta_2^4\theta_4^4-\theta_2^4\theta_3^4) = -\frac{1}{3}\,,\\
QC(A^{(4)}) &= \frac{1}{2^8 3}\frac{1}{\eta^{24}}\theta_2^8\theta_4^8(-\theta_3^4\theta_4^4-\theta_2^4\theta_3^4) = -\frac{1}{3}\,.
\end{aligned}\tag{4.12}$$

Where the following identities were used

$$\theta_2^4+\theta_4^4-\theta_3^4 = 0\,,\qquad \theta_2^4\theta_3^4\theta_4^4 = 16\eta^{12}\,,\qquad \theta_3^{12}-\theta_2^{12}-\theta_4^{12} = 48\eta^{12}\,. \tag{4.13}$$

Note that in (4.12) all dependence on powers of q^n with $n \neq 0$ has disappeared. With $c^{(1)}_{0,0} = -1$ and $c^{(2)}_{0,0} = c^{(3)}_{0,0} = c^{(4)}_{0,0} = 1/3$ the result for the non degenerate orbit is given by

$$I^{\mathrm{Tr}(F^4)}_{0,0} = -\frac{1}{3}\sum_N\sum_{N|n}\frac{1}{n}e^{-2\pi i NT} + \frac{2}{3}\sum_N\sum_{N|n}\frac{1}{n}e^{-2\pi i 2NT} + \text{c.c.} \tag{4.14}$$

4.3 $(tr(F^2)_1)^2$ thresholds

The operator Q for the $(tr(F^2)_1)^2$ threshold depending on the spin structures was defined in (3.6). Together with (3.4) $QC(A^{(i)})$ become

$$
\begin{aligned}
QC(A^{(1)}) &= \frac{1}{2^{10}3^2}\frac{1}{\eta^{24}}\Big\{\theta_2^{16}(e_2+\hat{E}_2)^2+\theta_3^{16}(e_3+\hat{E}_2)^2+\theta_4^{16}(e_4+\hat{E}_2)^2\Big\}\,,\\
QC(A^{(2)}) &= \frac{1}{2^{10}3^2}\frac{1}{\eta^{24}}\theta_2^8\theta_3^8\Big\{(e_2+\hat{E}_2)^2+(e_3+\hat{E}_2)^2\Big\}\,,\\
QC(A^{(3)}) &= \frac{1}{2^{10}3^2}\frac{1}{\eta^{24}}\theta_3^8\theta_4^8\Big\{(e_3+\hat{E}_2)^2+(e_4+\hat{E}_2)^2\Big\}\,,\\
QC(A^{(4)}) &= \frac{1}{2^{10}3^2}\frac{1}{\eta^{24}}\theta_2^8\theta_4^8\Big\{(e_2+\hat{E}_2)^2+(e_4+\hat{E}_2)^2\Big\}\,.
\end{aligned}
\tag{4.15}
$$

Exanding term in (4.15) it is easy to confirm that there are no terms of order $1/q$ present. Furthermore we get $c_{0,0}^{(1)}=1/8$,$c_{0,0}^{(2)}=c_{0,0}^{(4)}=1/4$ and $c_{0,0}^{(3)}=1/8$ and the result for the integral is then given by

$$
I=-\frac{1}{8}\sum_N\sum_{N|n}\frac{1}{n}e^{-2\pi i2NT}+\frac{1}{4}\sum_N\sum_{N|n}\frac{1}{n}e^{-2\pi iNT}+\text{c.c.} \tag{4.16}
$$

5. Type I′ Quantum mechanics

In order to determine the existence of bound states of $D0$ branes an index of the D0 brane QM has to be computed. In the case of type IIA D0 branes in ten dimensions this was done for the case of two D0 branes in [2, 3]. In this calculation the index is split into a bulk and boundary term which in turn is expressed as a zero dimensional matrix integral.

The hamiltonian for the $(0,8)$ quantum mechanics governing D0-particles in type I′ is given by (in the gauge $A_0=0$ and following the notation of [9])

$$
\begin{aligned}
H=&\frac{1}{2}\mathrm{Tr}\Big(\Pi_j^2-\Pi_\phi^2+g^2[\Phi,X_i]^2-\frac{g^2}{2}[X_i,X_j]^2\Big)+\frac{ig}{2}\mathrm{Tr}\Big(\lambda_{\dot\alpha}[\Phi,\lambda_{\dot\alpha}]+\Theta_\alpha[\Phi,\Theta_\alpha]-\\
&-2X_i\gamma^i_{\alpha\dot\alpha}\{\Theta_\alpha,\lambda_{\dot\alpha}\}\Big)-ig\Big(\chi_I^t\Phi\chi_I+m_{IJ}\chi_I^t\chi_J\Big).
\end{aligned}
\tag{5.1}
$$

All fields but the χ are given by an orientifold projection of the SU(N) D0-particle quantum mechanics, where X_i and Θ_α transform as the traceless symmetric representation of SO(N) which is given by the real matrices of the Lie algebra of SU(N). The spinor Θ_α transforms as 8_c spinor of SO(8) realated to the supersymmetries of the D0 brane unbroken by the presence of the D8 brane. In addition we have the trace part x_i and θ_α which are singlets under SO(N) and do not enter in the interacting hamiltonian (5.1). Φ and $\lambda_{\dot\alpha}$ transform under the adjoint representation of SO(N) which is given by the imaginary elements of SU(N) and transform as 1 and 8_s of SO(8) respectively. The chiral fermions χ_i^I transform in the real $(8,2N)$ of

SO(8)×SO(2N). Therefore giving nonzero values to the parameters m^{IJ} corresponds to moving the D8 branes away from the orientifold planes. The index calculated below will in principle depend on the values of the parameters m_{IJ}. In the following we will mostly be interested the case of all $m^{IJ}=0$ in the hamiltonian. The Gauss constraint is given by

$$G=[\Pi_j,X_j]-[\Pi_\phi,\Phi]+i\Theta_\alpha\Theta_\alpha-i\lambda_{\dot\alpha}\lambda_{\dot\alpha}+i\chi_I\chi_i^T\,. \tag{5.2}$$

The index of QM is given by

$$I_N=\lim_{\beta\to\infty}\mathrm{Tr}(-1)^F e^{-\beta H}\,, \tag{5.3}$$

where the trace is taken over gauge invariant states which satisfy $G=0$. An integration by parts turns the index into a bulk Z_N and deficit term δI_N, where $I=Z_N+\delta I_N$ and the bulk term is given by

$$Z_N=\lim_{\beta\to 0}\mathrm{Tr}(-1)^F e^{-\beta H}\,. \tag{5.4}$$

6. Type I′ interpretation

We want to use the results for the heterotic thresholds to determine matrix integrals of D0 particles in type I′ quantum mechanics. For simplicity we will consider a square torus with radii R_1,R_2 , the Kahler and complex structure moduli are then given respectively by

$$T=B_{12}^{NS}+iR_1R_2\,,\qquad U=i\frac{R_2}{R_1}\,. \tag{6.1}$$

Under heterotic type I duality the coupling constants, metric and AST field are related by

$$\lambda_{het}=\frac{1}{\lambda_I}\,,\qquad \lambda^I g_{\mu\nu}^{het}=g_{\mu\nu}^I\,,\qquad B_{\mu\nu}^{het}=B_{\mu\nu}^I\,. \tag{6.2}$$

Under a T-duality along the first circle type I gets mapped to type I′, where the radii are related by

$$R_1^I=\frac{1}{R^{I'}}\,,\qquad R_1\lambda^{I'}=\lambda^I\,,\qquad B_{12}^{RR,I}=A_2^{RR,I'}\,. \tag{6.3}$$

Hence the heterotic moduli T and U get mapped to

$$T=A_2^{RR}+i\frac{R_2'}{\lambda'}\,,\qquad U=iR_1'R_2'\,. \tag{6.4}$$

In the type I' variables the heterotic modulus T has the interpretation of the action of a Euclidean D0 brane worldline on a circle of radius R^2. On the other hand the modulus U is independent of the type I' coupling constant and can be interpreted as the action of open string worldsheet instantons which stretch between the two 8-brane/orientifold planes. In the limit of infinite separation of the 8-brane/orientifold planes $R_1 \to \infty$ all contributions of the form $\exp(2\pi k U)$ will therefore vanish for $k > 0$. For this limit to be meaninful it is important that terms of order $1/q$ in the integral (4.4) do not contribute as mentioned in section 4, since they will behave as $\exp(2\pi U_2)$ which diverges as $U_2 \to \infty$. Note that the in the limit $U_2 \to \infty$ the two O8 planes effectively decouple. A calculation as in section 4 for a $t_8 \mathrm{Tr}(F_1)^2 \mathrm{Tr}(F_2)^2$ threshold, where the two traces are over the two different SO(16), reveals that there are no terms which survive the $U_2 \to \infty$ limit. Hence the traces involving only one SO(16) factor should be sensitive only to the QM of D0 particles on one O8 plane. The situation for the gravitational threshold might be more complicated allthough the counting of fermionic zero modes suggests that the thresholds are only related to D0-branes on one of the two O8 planes. In the following we will identify $\exp(2\pi i N T)$ term in the threshold with the euclidean action for a worldline of a bound state of N D0 branes. The prefactor of the threshold should then be related to the bulk partition function Z_N of the index for N type I' D0-particles. This is the same idea used in [4] for the IIA D0 particle quantum mechanics. For odd N the $I_{0,0}$ calculated in (4.11), (4.14) and (4.16) are of the same form and it is tempting to conjecture that the value of Z_N up to an N independent numerical factor is given by

$$Z_N = const \sum_{N|n} \frac{1}{n}, \qquad N \text{ odd}. \tag{6.5}$$

The value of Z_N is determined up to an N independent constant which can in principle be determined by a careful analysis of the relative normalization of the heterotic calculation and the type I' QM. On the other had for even N, i.e. $N = 2N'$, the results for the integrals $I_{0,0}$ have a different structure than (6.5). The terms in $I_{0,0}$ (4.11), (4.14) and (4.16) proportional to $\exp(2\pi i 2N' T)$ are given by

$$Z_{2N'}^{tr(R^4)} = \frac{1}{2^6 3^2 5} \left(232 \sum_{N'|n} \frac{1}{n} + 256 \sum_{2N'|n} \frac{1}{n} \right), \tag{6.6}$$

and

$$Z_{2N'}^{tr(F^4)} = \frac{1}{3} \left(2 \sum_{N'|n} \frac{1}{n} - \sum_{2N'|n} \frac{1}{n} \right), \tag{6.7}$$

and

$$Z_{2N'}^{(tr(F^2))^2} = \frac{1}{8} \left(- \sum_{N'|n} \frac{1}{n} + 2 \sum_{2N'|n} \frac{1}{n} \right). \tag{6.8}$$

All these expression are of the form

$$Z_{2N'} = c_1 \sum_{N'|n} \frac{1}{n} + c_2 \sum_{2N'|n} \frac{1}{n}, \tag{6.9}$$

with some constants c_1 and c_2 and it is natural to assume that the bulk part of the index has the same structure, allthough it is at present not clear whether one can read off the value of the constants from (6.6) directly.

A possible explanation for this behavior of the thresholds for even N could be that the heterotic threshold corrections are not related directly to the bulk part of the index but to some correlation function for the QM, which differs from the bulk index for $2N$ D0 particles but is proportional to it for odd number of $D0$ particles.

The fields entering the quantum mechanics also contain eight singlet bosons x_i, $i = 1, \ldots, 8$ and eight fermions $\theta_a, a = 1, \ldots, 8$. The R^4 threshold then corresponds to the loop amplitude of a D0 brane coupling to four gravitons. The vertex operator for a graviton with polarization ternsor h_{ij} coupling to the D0 branes is given by

$$V(h) = h_{ij} k_k \theta^\alpha \gamma^{ik}_{\alpha\beta} \theta^\beta \dot{X}^j e^{ikX} \,. \tag{6.10}$$

The insertions of the four graviton vertices soaks up the eight fermionic zero modes θ_a. The SO(N) part of the QM does not couple to these 'center of mass' coordinates and hence the R^4 threshold should then be multiplied by a partition function of the SO(N) degrees of freedom which we interpret as the bulk term for the QM.

In the case of $\mathrm{Tr}(F^4)$ and $(\mathrm{Tr}(F^2))^2$, the situation is more complicated since the gauge fields live on the D8 brane and there is a coupling between these and the SO(N) QM via χ^a_I. A vertex for a gauge field with field strength F^{IJ}_{ij} for a D0 brane is of the form

$$V(F) = F^{IJ}_{ij} \theta^\alpha \gamma^{ij}_{\alpha\beta} \theta^\beta \chi^I_a \chi^J_a e^{ikX} \,. \tag{6.11}$$

Hence a D0-brane loop coupling to four gauge fields will correspond to the insertion of four such vertex operators, which again soak up the eight θ zero modes. The insertion of $\chi^J_a \chi^J_a$ in the path integral of the SO(N) is equivalent to taking derivatives $\partial/\partial m_{IJ}$ of Z_N in (5.4). The results of the heterotic threshold corrections predict these correlation functions, if this interpretation is correct.

$$\begin{aligned} \left(\mathrm{Tr}(F^2)\right)^2 : &\quad \frac{\partial}{\partial m_{IJ}} \frac{\partial}{\partial m_{IJ}} \frac{\partial}{\partial m_{KL}} \frac{\partial}{\partial m_{KL}} Z_N \bigg|_{m_{AB}=0} \,, \\ \mathrm{Tr}(F^4) : &\quad \frac{\partial}{\partial m_{IJ}} \frac{\partial}{\partial m_{JK}} \frac{\partial}{\partial m_{KL}} \frac{\partial}{\partial m_{LI}} Z_N \bigg|_{m_{AB}=0} \,. \end{aligned} \tag{6.12}$$

In particular the heterotic threshold calculation implies that there is a difference between the case N even and N odd for these correlation functions. It would be interesting to check this conjectured result explicitly.

7. D0-particle loop

There is a simple picture of the result for the bulk part of the index Z_N in (6.5). N D0 particles will form bound states which transform (according to heterotic type I duality) as the **128** of SO(16) for odd N and as the **120** of SO(16) for even N. One

JHEP05(1999)007

can imagine that the D0 particles are stuck on the D8-O8 branes. The $\mathrm{Tr}(F_1)^4$ can then be interpreted as coming from a loop of D0 particles with four graviton vertex operators (6.10) inserted.

$$I = \sum_n \frac{1}{\pi^{D/2}} \int d^D p \int \frac{dt}{t} t^k \exp\left(-t\left(p^2 + \mu^2 - \frac{(n-A)^2}{R^2}\right)\right), \tag{7.1}$$

where μ is the mass of a D-particle. If the D-particle is stuck on the D8-brane the momentum integral is nine dimensional, i.e. $D = 9$. Furthermore inserting four vertices to soak up fermionic zero modes introduces a factor of t^4, i.e. $k = 4$ in (7.1). After integrating out the loop momentum and performing a Poisson resummation over n, we get

$$\begin{aligned} I &= R\sqrt{\pi} \sum_m \int \frac{dt}{t^{\frac{3}{2}}} \exp\left(-\frac{\pi^2 R^2 m^2}{t} - \mu^2 t + 2\pi i m A\right) \\ &= \sum_m \frac{1}{m} \exp\left(-2\pi R m \mu + 2\pi i m A\right). \end{aligned} \tag{7.2}$$

In the second line formula (A.3) from the appendix has been used. The mass μ of a D0 particle of charge n given by $\mu = n/\lambda$ and $A = nA^{RR}$. Hence summming over the contribution of a charge n D0 particle winding m times gives

$$Z = \sum_{m,n} \frac{1}{m} e^{-2\pi m n R/\lambda + 2\pi i m n A} = \sum_N \sum_{N|n} \frac{1}{n} e^{-2\pi N R/\lambda + 2\pi i N A}. \tag{7.3}$$

Note that there is a difference to the case of the IIB D0 particle analysis given in [4]. Due to the fact that the momentum integral is only nine dimensional (since we assumed that the D0 brane is stuck on the orientifold plane) the integral reduces to a Bessel function $K_{1/2}$ instead of K_1. Since the series expansion for $K_{1/2}$ terminates after one term this implies that there is no infinite asymptotic series of corrections. This behavior should reflect the exact cancellation of bosonic and fermionic fluctuations for the type I' quantum mechanics. Note also that the extra contribution for the $\mathrm{Tr}(F)^4$ and $(\mathrm{Tr}(F^2))^2$ thresholds for $N = 2N'$ could be interpreted as coming from N' D0 particles with charge two. This might come from D0 particle pairs which move pairwise off the D8 plane and form a bound state with twice the charge.

8. Conclusions

In this note a heterotic one loop calculation of threshold corrections in the presence of Wilson lines was performed. Using the heterotic type I duality and the T-duality relating type I and type I′ it was argued that from these thresholds information about certain quantities calculated in type I′ quantum mechanics can be extracted. At the moment the status of this claim is not certain. A puzzling feature is the difference in the structure of Z_N calculated in section 4 for even and odd number of D0 particles. In particular for the gravitational threshold $I_{0,0}^{\mathrm{Tr}(R^4)}$ seems unlikely to be directly related to the bulk term for even N. One (disappointing) possibility is that the quantities calculated in this note are not directly related to the bulk terms of the index (or correlation functions) for type I′ QM. On the other hand it would be very interesting to adress this question by a direct calculation of in type I′ QM for arbitary N. This seems to be a very difficult task. Another interesting question would be to consider more general Wilson lines than (3.1) corresponding to moving the D8 branes off the orientifold planes [9, 10]. In principle the calculation in this note can easily be generalized to the more general case. In addition it is not clear wether the part of the threshold which depend on $\exp(2\pi iU)$ have an interpretation in the D0-brane quantum mechanics.

Acknowledgments

I wish to thank Piljin Yi for useful correspondences and comments. I am grateful to Stephan Stieberger for pointing out an error in a previous version of the paper. This work was supported in part by NSF grant PHY-9802484.

A. Evaluation of the integrals

In this appendix we review the evaluation of integrals appearing in the heterotic threshold calculation. The basic technique was developed in [27] for more details in this context see [19, 20].

$$I_{n,r} = \sum_{\substack{k>0\\0\le j<k\\p\neq 0}} e^{2\pi ikpT} \int \frac{d^2\tau}{\tau_2^2} \exp\left(\frac{\pi T_2}{\tau_2 U_2}|k\tau + j + pU|^2\right) \frac{1}{\tau_2^r} \exp(2\pi i\tau n)\,. \tag{A.1}$$

Integrating over τ_1 gives

$$I_{n,r} = \sum_{\substack{k>0\\0\le j<k\\p\neq 0}} \frac{\sqrt{U_2}}{k\sqrt{T_2}} e^{2\pi ikpT} e^{2\pi in(j+pU_1)/k+2\pi kpT_2} \int \frac{d\tau_2}{\tau_2^{3/2+r}} e^{-\frac{\pi T_2}{U_2}\left(k+\frac{nU_2}{kT_2}\right)^2\tau_2} e^{-\pi p^2 T_2 U_2/\tau_2}\,. \tag{A.2}$$

The integral over τ_2 can be done using the formula

$$\int_0^\infty \frac{dx}{x^{3/2+r}} e^{-ax-b/x} = \left(-\frac{d}{db}\right)^r \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}}\,, \tag{A.3}$$

where

$$a = \frac{\pi T_2}{U_2}\left(k + \frac{nU_2}{kT_2}\right)^2, \qquad b = \pi p^2 T_2 U_2\,. \tag{A.4}$$

We are primarily interested in the evaluation of the integrals in the large U_2 limit. it is easy to see that the leading contribution in (A.3) is obtained when all the derivatives act on the exponential in (A.3). Since $a = \pi n^2 U_2/(T_2 k^2) + o(1)$ all the integrals $I_{0,r}$ will be suppressed by factors of $1/U_2^r$ for $r > 0$, because the leading term is proportional to n, which vanishes for $n = 0$. The final result for for the leading U_2 independent term in $I_{0,0}$ is the given by

$$I_{0,0} = \sum_j \sum_{k>0,p>0} \frac{1}{k|p|} e^{2\pi i k p T} + \text{c.c.}\,, \tag{A.5}$$

where in applications of this formula in section 4 the sumation range of j depends on the sector $A^{(i)}$ which is considered.

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JHEP05(1999)007

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